Circular vs triangular cross sections: some thoughts about bending stiffness

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Abstract

Ski equipment producer SWIX has recently presented a new pair of ski poles, called SWIX Triac, which differs from conventional (round) ski poles by having a triangular cross section. SWIX claims that the main objective for this design is that it has superior stiffness to weight ratio compared to common ski poles. We prove in this paper that this claim in general is not true. More specific, we show that for thin walled cross sections, a hollow circular cross section has up to 36% better stiffness to weight ratio than a corresponding triangular cross section.

1 Motivation for this paper

The inspiration to this paper was some statements we found in a PDF document (see [5]) on the web page of ski equipment producer SWIX (`http://www.swixsport.com`). More specific, this document contained some facts about their recently designed ski poles SWIX Triac, which in contrast to conventional (round) ski poles have a triangular cross section. Many of the top cross country skiers in the world today use SWIX Triac ski poles, see Figure 1 below.

Figure 1: Norwegian cross country skier Anders Gløersen in Otepää, Estonia, 2010.

One of SWIX main objectives for this choice of design is found in the following statement: *"Weight, stiffness and strength are dictated by the amount of material used and the circumference of the shaft. A* *triangle encircled by a circle has a smaller circumference than the circle. By using a triangle geometry less material is needed and weight is saved. The distance from the centre of the shaft, to the outer edge of the shaft is the dimension that gives a pole shaft its stiffness. A triangle has the same distance from the centre to a corner as the circle has although only in three directions. By orienting the corners in optimum directions we can achieve the same geometry stiffness (if not increase) with a reduced amount of material and thus decreased overall weight of the shaft."*

The conclusions in this statement does not coincide with what our engineering intuition tells us, thus the goal with this paper is to prove that the statement above in general is not true.

2 Introduction

A long and slender construction element (called a strut) is likely to fail in (Euler) buckling during high compressive loading. Euler buckling is caused by elastic instability of the strut, where the actual compressive stress is lower than the ultimate compressive stress the material can handle.

The maximum (compressive) axial load a long, slender strut can carry without buckling is called the critical load, F_{cr}. This load causes the strut to be in a state of unstable equilibrium, where the slightest lateral load will make the strut to fail by buckling. The formula for the critical load of a straight homogeneous column was found in 1757 by Swiss mathematician Leonhard Euler and states that

$$F_{cr} = \frac{\pi^2 EI}{K^2 l^2},$$

where l is the length of the column, E is Young's modulus of the material, I is the area moment of inertia (in the direction where it has the lowest value) and K is a constant depending on the type of support at the ends.

Assuming that the length is fixed and that we have already chosen one material (fixed E), the only method we have to increase the critical load is to choose a suitable cross section with large area moment of inertia I, given by

$$I = \iint\limits_S y^2 dx\ dy,$$

where S is the cross section. Assume now that we want to compare two different cross sections with the same area A (and hence the same total mass)

$$A = area(S) = \iint\limits_S dx\ dy.$$

In order to determine the effectiveness of a given cross section, we therefore introduce a so called shape factor ϕ, which is a dimensionless parameter which compares the area moment of inertia I of our cross section with that of some given pre-determined simple cross section I_0, under the assumption of equal cross section area $A = A_0$. To this end, let us take a solid circular cross section with radius r as our simple cross section S_0. Then the cross section area A_0 and area moment of inertia I_0 are

$$\begin{aligned} A_0 &= \pi r^2, \\ I_0 &= \frac{\pi r^4}{4} = \frac{A_0^2}{4\pi}. \end{aligned}$$

We now define the shape factor ϕ for a given cross section S to be the ratio

$$\phi = \frac{I}{I_0}$$

under the assumption of equal cross section area $A = A_0$, which implies that

$$\phi = \frac{I}{\frac{A_0^2}{4\pi}} = \frac{4\pi I}{A^2}.$$

Thus, if a given cross section has a shape factor of for instance $\phi = 4$, it means that it can withstand 4 times higher compressive force without buckling than a solid circular cross section with the same cross section area.

Let us first compare the shape factor of a solid triangular cross section. Then we will compute approximate shape factors for thin-walled cross sections (under the assumption that the thicknesses are much lower than the overall dimension) and after that compare with a numerical example. Finally, we will compute the exact expressions for the cross sections and from that conclude that a hollow triangular cross section is inferior to a hollow circular cross section when one wants to maximize the critical force without increasing the mass, when the thickness is relatively small compared to the overall dimension. For readers interested in theory of buckling and beam bending, we refer to any of the classic books written by Stephen Timoshenko, for instance [3] or [4] or the classic book by Jones [2].

3 Strut buckling

It is reasonable to assume the ski pole is compressed by loads F that are applied with a small eccentricity e measured from the axis of the strut. This gives rise to a centric load F and a moment couple $M_0 = Fe$. This moment causes the strut to deflect $-w(z)$ at the onset of loading. Therefore, the bending moment M in the strut at height z over the lower end is

$$M(z) = M_0 + F(-w) = F(e - w(z)).$$

This bending moment will curve the ski poles with a curvature κ that is proportional to the applied moment M, with the proportional coefficent being the reciprocal of the product of Young's modulus E and the area moment of inertia I (EI is called the *bending stiffness*), i.e.

$$\kappa = \frac{d^2w}{dz^2} = \frac{M}{EI} = \frac{F(e - w(z))}{EI}.$$

That is, the differential equation of the deflection curve is

$$w'' + \lambda^2 w = \lambda^2 e,$$

where

$$\lambda = \sqrt{\frac{F}{EI}}.$$

This equation has the general solution

$$w(z) = C \sin \lambda z + D \cos \lambda z + e,$$

in which A and B are constants of integration. It is natural to assume that the ends of the ski pole act as pin joints, giving the boundary conditions

$$\begin{aligned} w(0) &= D + e = 0 \\ w(l) &= C \sin \lambda l + D \cos \lambda l + e = 0. \end{aligned}$$

Hence

$$\begin{aligned} C &= -\frac{e\left(1-\cos\lambda l\right)}{\sin\lambda l} = -e\tan\frac{\lambda l}{2}, \\ D &= -e, \end{aligned}$$

giving

$$w\left(z\right) = e\left(1-\tan\frac{\lambda l}{2}\sin\lambda z - \cos\lambda z\right).$$

This gives the corresponding bending moment

$$M\left(z\right) = F\left(e - w\left(z\right)\right) = Fe\left(\tan\frac{\lambda l}{2}\sin\lambda z + \cos\lambda z\right).$$

To find the maximum deflection δ, we take the derivative of this expression and find the value of z for which it is zero, that is

$$w'\left(z\right) = e\lambda\left(\sin\lambda z - \tan\frac{\lambda l}{2}\cos\lambda z\right) = 0,$$

which implies that

$$\tan\lambda z = \frac{\sin\lambda z}{\cos\lambda z} = \tan\frac{\lambda l}{2}.$$

Thus the maximum deflection δ produced by the eccentric load occurs at the midpoint of the column $z = l/2$ and is

$$\delta = -w\left(\frac{l}{2}\right) = e\left(\tan\frac{\lambda l}{2}\sin\frac{\lambda l}{2} + \cos\frac{\lambda l}{2} - 1\right) = e\left(\frac{1}{\cos\frac{\lambda l}{2}} - 1\right).$$

The corresponding maximum moment along the strut is then

$$M_{\max} = M\left(\frac{l}{2}\right) = \frac{Fe}{\cos\frac{\lambda l}{2}}.$$

We see that the maximum deflection becomes large when

$$\lambda \to \lambda_{cr} = \frac{\pi}{l}$$

from below. The so called critical load F_{cr} is then the limit

$$F_{cr} = EI\lambda_{cr}^2 = \frac{\pi^2 EI}{l^2}.$$

Let us now look for the maximum stress in the strut. It will occur at the cross section where the deflection and bending moments have their largest values, that is, at the midpoint $z = l/2$. Acting at this cross section are the compressive force F and the bending moment $M_{\max}$, giving the stresses F/A and $M_{\max}y_{\max}/I$, respectively, where A is the cross section area and $y_{\max}$ the distance from the centroidal axis to extreme point on the concave side of the strut. Thus the total stress is

$$\sigma = \frac{F}{A} + \frac{M_{\max}y_{\max}}{I} = \frac{F}{A}\left(1 + \frac{Ay_{\max}}{I}\frac{e}{\cos\frac{\lambda l}{2}}\right).$$

4 Shape factor of a solid triangular cross section

First, let us compare a solid equilateral triangular cross section to a solid circular cross section. Let L be the side of the triangle. Then the triangular cross section, called S_T, has cross section area and area moment of inertia (see for instance [1])

$$\begin{aligned} A_T &= area(S_T) = \frac{\sqrt{3}L^2}{4}, \\ I_T &= \frac{L^4}{32\sqrt{3}}, \end{aligned}$$

giving a shape factor of

$$\phi_T = \frac{4\pi I_T}{A_T^2} = \frac{4\pi \frac{L^4}{32\sqrt{3}}}{\left(\frac{\sqrt{3}L^2}{4}\right)^2} = \frac{2\pi}{3\sqrt{3}} \approx 1.21.$$

Hence a solid triangular cross section is 21% stiffer than a solid circular cross section with the same cross section area.

Remark 1 *The area moment of inertia for circular and equilateral triangular cross sections are equal in all directions. This follows since $I_{xx} = I_{yy}$ and $I_{xy} = 0$, where the last identity follows from integration of an odd function over a symmetric region. Assume now that we rotate the coordinate system the angle θ. Let us call the rotated x-axis ξ and the rotated y-axis η. Then the area moments of inertia in the new coordinate system follow from the expressions*

$$\begin{aligned} I_{\xi\xi} &= \frac{1}{2}(I_{xx} + I_{yy}) + \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta - I_{xy}\sin 2\theta = I_{xx}, \\ I_{\eta\eta} &= \frac{1}{2}(I_{xx} + I_{yy}) - \frac{1}{2}(I_{xx} - I_{yy})\cos 2\theta + I_{xy}\sin 2\theta = I_{xx}, \\ I_{\xi\eta} &= \frac{1}{2}(I_{xx} - I_{yy})\sin 2\theta + I_{xy}\cos 2\theta = 0. \end{aligned}$$

In other words, profiles having circular or equilateral triangular cross sections are equally stiff in all directions. The same conclusion is also true for hollow circular or equilateral triangular cross sections.

5 Comparison of two different thin-walled hollow cross sections

Let us now compare the effectiveness of two different hollow cross sections. Let the first cross section S_R be a hollow circular cross section with mean radius

$$r = \frac{R_o + R_i}{2}$$

and thickness

$$t = R_o - R_i \ll r,$$

where R_o and R_i denote outer and inner radii, respectively. Let the second cross section S_T be a hollow equilateral triangular cross section with mean side

$$L = \frac{L_o + L_i}{2}$$

and same thickness

$$t = \frac{L_o - L_i}{2\sqrt{3}} \ll L,$$

where L_o and L_i denote outer and inner sides, respectively. Note that we assume that the thickness t is same for both cross sections, see Figure 1 below.

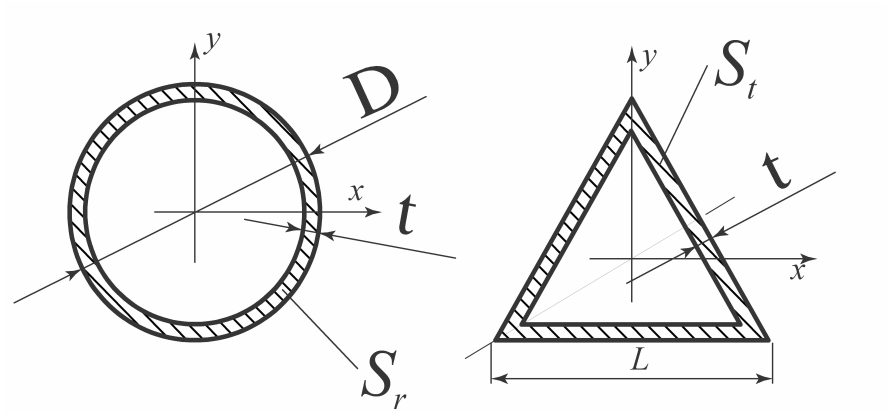

Figure 2: Two hollow cross sections.

Then the cross sections have areas

$$\begin{aligned} A_R &= area(S_R) = \pi R_o^2 - \pi R_i^2 = \pi\left[\left(r + \frac{t}{2}\right)^2 - \left(r - \frac{t}{2}\right)^2\right] = 2\pi r t, \\ A_T &= area(S_T) = \frac{\sqrt{3}L_o^2}{4} - \frac{\sqrt{3}L_i^2}{4} = \frac{\sqrt{3}}{4}\left[\left(L + \sqrt{3}t\right)^2 - \left(L - \sqrt{3}t\right)^2\right] = 3Lt, \end{aligned}$$

and area moments of inertia

$$\begin{aligned} I_R &= \pi\frac{R_o^4}{4} - \pi\frac{R_i^4}{4} = \frac{\pi}{4}\left[\left(r + \frac{t}{2}\right)^4 - \left(r - \frac{t}{2}\right)^4\right] = \frac{\pi r t\left(4r^2 + t^2\right)}{4} \approx \pi r^3 t, \\ I_T &= \frac{L_o^4}{32\sqrt{3}} - \frac{L_i^4}{32\sqrt{3}} = \frac{1}{32\sqrt{3}}\left[\left(L + \sqrt{3}t\right)^4 - \left(L - \sqrt{3}t\right)^4\right] = \frac{Lt\left(L^2 + 3t^2\right)}{4} \approx \frac{L^3 t}{4}. \end{aligned}$$

This gives approximate shape factors for the thin-walled hollow circular and triangular cross sections of

$$\begin{aligned} \phi_R &= \frac{4\pi I_R}{A_R^2} = \frac{r}{t}, \\ \phi_T &= \frac{4\pi I_T}{A_T^2} = \frac{\pi L}{9t}. \end{aligned}$$

Since we want to know which cross section is most efficient, we assume that we have the same cross section area $A_R = A_T$, giving

$$L = \frac{2\pi r}{3}.$$

Hence, expressed in terms of only r and t, the approximate shape factors for equal cross section areas are

$$\begin{aligned}\phi_R &= \frac{r}{t},\\ \phi_T &= \frac{2\pi^2}{27}\frac{r}{t}.\end{aligned}$$

Since

$$\frac{\phi_R}{\phi_T} = \frac{27}{2\pi^2} \approx 1.3678,$$

we can conclude that the thin walled circular cross section is approximately $36 - 37\%$ stiffer than the triangular cross section with equal cross section area, regardless of the actual dimensions. Note that we have made a small approximation when we discarded the term containing t^2, since if for instance $t/L < 0.1$, we get that

$$L^2 < L^2 + 3t^2 < 1.03L^2,$$

implying that the approximation $L^2 + 3t^2 \approx L^2$ is valid and analogously for the factor $4r^2 + t^2$.

6 Induced stresses

We saw in the introduction that if we compress the pole with a given force F, the induced stress is

$$\sigma = \frac{F}{A} + \frac{M_{\max} y_{\max}}{I} = \frac{F}{A}\left(1 + \frac{A y_{\max}}{I}\frac{e}{\cos\frac{\lambda l}{2}}\right),$$

where

$$\lambda = \sqrt{\frac{F}{EI}}$$

Let us first consider the cosine factor. Since cosine is locally decreasing when its argument is increasing from zero, we have that

$$I_1 > I_2 \Leftrightarrow \lambda_1 < \lambda_2 \Leftrightarrow \frac{1}{\cos\frac{\lambda_1 l}{2}} < \frac{1}{\cos\frac{\lambda_2 l}{2}}.$$

Therefore, a larger value of I will make the contribution from the cosine factor to the stress smaller.

Let us first assume that the contribution from the cosine factor is independent of I. Again, we assume that we have the same cross section $A_R = A_T$. Then the induced stress depends on the quotient

$$Z = \frac{y_{\max}}{I},$$

which is

$$Z_R = \frac{r}{I_R} = \frac{1}{\pi r^2 t}$$

and

$$Z_T = \frac{\frac{2}{3}\frac{\sqrt{3}}{2}L}{I_T} = \frac{4}{\sqrt{3}L^2 t},$$

since two thirds of the triangular height is the maximal distance from the centroidal axis to the outer surface for the triangular cross section. Hence, using

$$L = \frac{2\pi r}{3},$$

we get that

$$\frac{Z_R}{Z_T} = \frac{\sqrt{3}L^2}{4\pi r^2} = \frac{\pi}{3\sqrt{3}} \approx 0.60.$$

If we now also include the fact that the area moment of inertia for a thin walled circular cross section is larger than for the corresponding triangular cross section, the term

$$\frac{Ay_{\max}}{I} \frac{e}{\cos \frac{\lambda l}{2}}$$

in the expression for the induced stress in the circular cross section is less than 60% of the corrsponding value for a thin walled triangular section.

All in all, the stress induced in a compressed ski pole is significantly lower in a pole with circular cross section than in a ski pole with triangular cross section.

7 Comparison with some numerical values

Let us compare the approximate results above with some exact numerical values. As an example, we let the outer radius be $R_0 = 8.0$ mm and the inner radius be $R_i = 7.0$ mm. This gives a mean radius of $r = 7.5$ mm and thickness $t = 1.0$ mm. Then the mean triangle side is

$$L = \frac{2\pi r}{3} = 15.71\text{mm}$$

giving outer and inner sides of

$$\begin{aligned} L_o &= L + \sqrt{3}t = 17.44 \text{ mm}, \\ L_i &= L - \sqrt{3}t = 13.98 \text{ mm}. \end{aligned}$$

Then the circular and triangular cross section areas are

$$A_R = A_T = 47.12 \text{ mm}^2$$

and the area moments of inertia

$$\begin{aligned} I_R &= \frac{\pi}{4}\left(R_o^4 - R_i^4\right) = 1331.2 \text{ mm}^4, \\ I_T &= \frac{1}{32\sqrt{3}}\left(L_o^4 - L_i^4\right) = 980.7 \text{ mm}^4, \end{aligned}$$

respectively. Hence the circular cross section is

$$\frac{1331.2}{980.7} \approx 1.36$$

times stiffer than the triangular cross section.

Remark 2 *In order to determine how large error one makes when doing the approximation in the previous section, we can compare the approximate values for the thin walled cross sections with the exact values calculated in the example above. The approximate values are*

$$\begin{aligned} I_R^* &= \pi r^3 t = 1325.4 \ mm^4, \\ I_T^* &= \frac{L^3 t}{4} = 968.95 \ mm^4, \end{aligned}$$

which is about 1% off the correct values.

8 Exact values for the shape factors

We saw that while a solid equilateral triangular cross section was better than a solid circular cross section, the opposite was true for thin walled hollow cross sections.

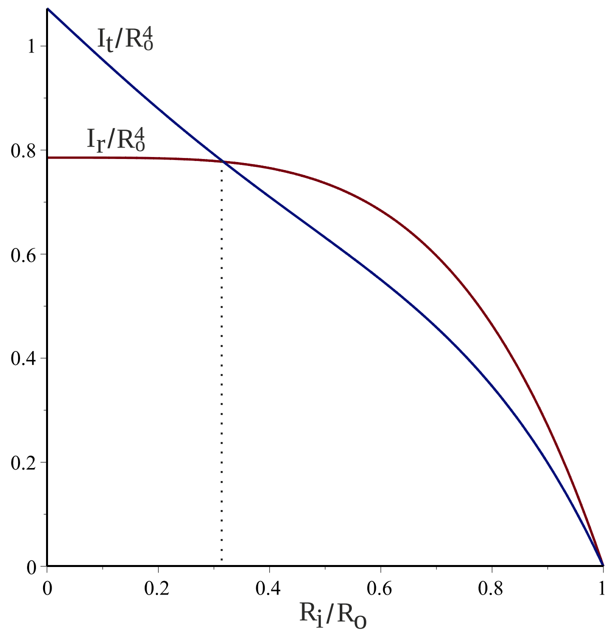

Figure 3: I_R and I_T (normalized by R_o^4) plotted against the ratio R_i/R_o.

Thus we realize that there must be some point in between these extremes, where a hollow circular cross section is equally effective as a hollow triangular cross section, see Figure 2 below where we have plotted I_R and I_T (normalized by R_o^4) against the ratio R_i/R_o. Let us look for this point. We had that

$$\begin{aligned} I_R &= \frac{\pi r t \left(4r^2 + t^2\right)}{4}, \\ I_T &= \frac{Lt \left(L^2 + 3t^2\right)}{4}, \end{aligned}$$

when

$$L = \frac{2\pi r}{3}$$

for equal cross section area. This gives the ratio

$$\frac{I_R}{I_T} = \frac{\pi r \left(4r^2 + t^2\right)}{L\left(L^2 + 3t^2\right)} = \frac{3\left(4r^2 + t^2\right)}{2\left(\left(\frac{2\pi r}{3}\right)^2 + 3t^2\right)} = \frac{27}{2\pi^2} \frac{1 + \frac{1}{4}\left(\frac{t}{r}\right)^2}{1 + \frac{27}{4\pi^2}\left(\frac{t}{r}\right)^2}.$$

Let us therefore check the behavior of the function

$$f\left(x\right) = \frac{27}{2\pi^2} \frac{1 + \frac{1}{4}x^2}{1 + \frac{27}{4\pi^2}x^2}.$$

First, we see that

$$f(0) = \frac{27}{2\pi^2} \approx 1.36$$

which is the limit in the case when the thickness goes to zero. Next, let us try to solve the equation

$$f(x) = 1, \; x > 0,$$

which gives the point where a hollow circular and equilateral triangular cross sections are equally effective, see Figure 3 below.

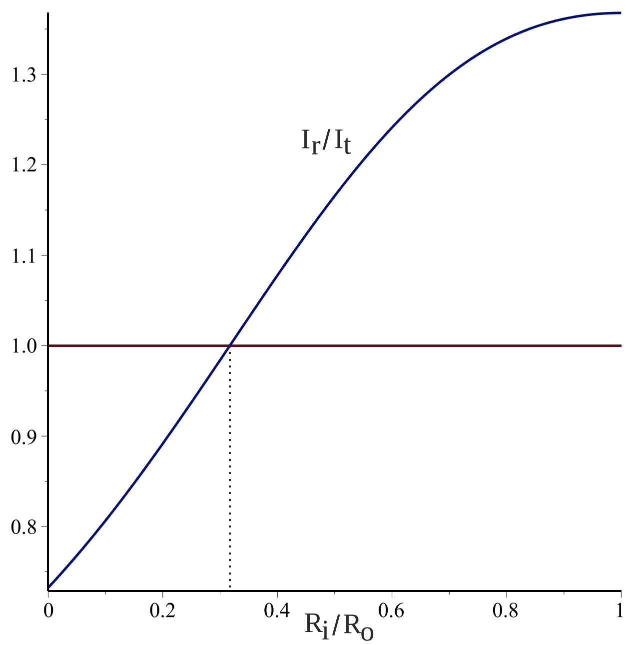

Figure 4: I_R/I_T plotted against R_i/R_o. Intersection with 1 gives the point where hollow circular and triangular cross sections are equally effective.

In this point, we must have

$$27\left(1 + \frac{1}{4}x^2\right) = 2\pi^2\left(1 + \frac{27}{4\pi^2}x^2\right),$$

that is,

$$x = \frac{t}{r} = \sqrt{4 - \frac{8\pi^2}{27}} \approx 1.037.$$

Hence a hollow circular cross section is more efficient than a hollow equilateral triangular cross section when the thickness t is less than 1.037 times the mean radius r and opposite when the thickness is larger than 1.037 times the mean radius. Since

$$\begin{aligned} R_o &= r + \frac{t}{2}, \\ R_i &= r - \frac{t}{2}, \end{aligned}$$

we have that

$$\frac{R_i}{R_o} = \frac{2 - \frac{t}{r}}{2 + \frac{t}{r}} = \frac{27 - 3\sqrt{81 - 6\pi^2}}{\pi^2} - 1 \approx 0.317$$

in this case. Summing up, a hollow circular cross section is more efficient than a hollow equilateral triangular cross section when the ratio between the inner radius and the outer radius is larger than 0.317 and opposite when the ratio is less than 0.317.

9 Conclusion

We have shown that a circular hollow cross section is stiffer than a triangular hollow cross section with the same cross section area when inner radius is larger than about one third of the outer radius. It therefore does not seems to be correct from the point of optimizing bending stiffness to choose a triangular cross section over a circular cross section for thin walled structures. However, there might be other physical reasons for such a choice, not covered here or discussed in the document published by SWIX [5].

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